Formal preliminaries

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September 14, 2015

1 Today

Learn and/or review some basic formal concepts, starting with basic set theory and building up to relations and functions.

It's important to do this. These formal tools are the bedrock of our theory of interpretation and basic meanings.

2 Sets

Set: "a collection of distinct objects, considered as an object in its own right" (thx wikipedia!).

** Note the restriction to distinct objects. $\{1,3,5,6,1,3\}$ is coherent, but not any different from $\{1,3,5,6\}$.

The objects in a set are called its **members**. They can be anything (people, plants, numbers, animals, ...). A lot of our examples will use numbers because they're easy to think about, but that's not essential.

Ways to define a set (notice I cut the pie differently from H&K here):

i. **Extensionally**: by listing elements. E.g. $\{1,2,3\}$, $\{\textsf{bob},\textsf{sue},\textsf{uni}\}$, ...
ii. **Intensionally**: by appeal to some properties the members of the set have to meet. Makes it possible to define an infinite set. E.g. the set of all natural numbers, $\{x : x \text{ is a person in this room}\}$, ...

Some more examples:

i. $A := \{n : n \text{ is an integer, and } 1 \leq n \leq 5\}$
ii. $B := \{c : c \text{ is a color of the French flag}\}$
iii. $C := \{n^2 - 4 : n \text{ is an integer, and } 0 < n \leq 8\}$

The empty set: the unique set which has no members. Variously written $\emptyset$, $\{\}$. E.g. $\{x : x \text{ is both even and odd}\}$.

Preview: we will use sets to characterize the meanings of common nouns like *bachelor*, intransitive verbs like *left*, and adjectives like *blue*.

2.1 Relations between sets

Subset: $A \subseteq B$ iff every member of A is also a member of B. Danger! Some authors (e.g. Allwood et al., alas) use $A \subset B$ to mean the same thing as $A \subseteq B$. This is pretty unusual, and we won't use it. Instead we'll reserve $\subset$ for....

Proper subset: $A \subset B$ iff $A \subseteq B$, and $B \not\subseteq A$.

Equivalence: $A = B$ iff $A \subseteq B$, and $B \subseteq A$.

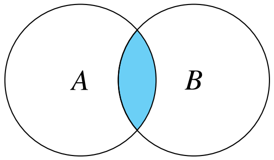
Superset: $B \supseteq A$ iff $A \subseteq B$

Examples:

i. $\{1,2,3\} \subset \{1,2,3\}$?
ii. $\{x : x \text{ is a tall woman}\} \subseteq \{x : x \text{ is a woman}\}$?
iii. $\emptyset \subseteq \emptyset$?

2.2 Operations on sets

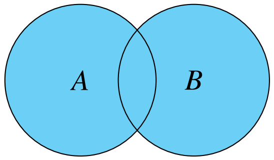
Intersection: $A \cap B := \{x : x \in A \text{ and } x \in B\}$



Examples:

i. $\{1,2,3,4,5\} \cap \{4,5,6,7\} =$
ii. $\{3,5\} \cap \{3,5\} =$
iii. $\{4,7,8\} \cap \emptyset =$
iv. $\{x : x \text{ is an odd number}\} \cap \{x : x \text{ is an even number}\} =$

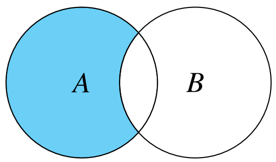
Union: $A \cup B := \{x : x \in A \text{ or } x \in B\}$



Examples:

i. $\{1,2,3,4,5\} \cup \{4,5,6,7\} =$
ii. $\{1,2,3,4,5\} \cup \{1,2\} =$
iii. $\{4,7,8\} \cup \emptyset =$
iv. $\{x : x \text{ is an odd number}\} \cup \{x : x \text{ is an even number}\} =$

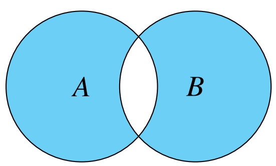
Difference: $A - B := \{x : x \in A \text{ and } x \notin B\}$



Examples:

i. $\{1,2,3,4,5\} - \{4,5,6,7\} =$
ii. $\{1,2,3,4,5\} - \{1,2\} =$
iii. $\{4,7,8\} - \emptyset =$
iv. $\{x : x \text{ is an odd number}\} - \{x : x \text{ is an even number}\} =$

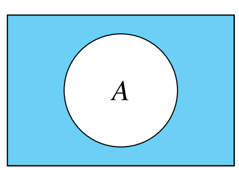
Exclusive union: $A \underline{\cup} B := (A \cup B) - (A \cap B)$



Examples:

i. $\{1,2,3,4,5\} \underline{\cup} \{4,5,6,7\} =$
ii. $\{1,2,3,4,5\} \underline{\cup} \{1,2\} =$
iii. $\{4,7,8\} \underline{\cup} \emptyset =$
iv. $\{x : x \text{ is an odd number}\} \underline{\cup} \{x : x \text{ is an even number}\} =$

Complementation: $\bar{A} := \{x : x \notin A\}$



Notice that it only makes sense to talk about the complement of a set relative to some "universe of discourse". Given a *universe* of discourse U, complementation is equivalently given by $U - A$.

Examples:

i. Given $U := \{x : x \text{ is an integer}\}$, $\overline{\{x : x \text{ is odd}\}} =$
ii. Given $U := \{x : x \text{ is a man}\} \cup \{x : x \text{ is a cat}\}$, $\overline{\{x : x \text{ is a man}\}} =$

Powerset: given a set A, the powerset of A is the set of all subsets of A. More formally $\wp(A) := \{B : B \subseteq A\}$. Also written as 2^A (why?).

Examples:

i. $\wp(\emptyset) =$
ii. $\wp(\{1\}) =$
iii. $\wp(\{1,2\}) =$
iv. $\wp(\{1,2,3\}) =$

2.3 Woe is u

Sets of sets are perfectly coherent (and, it turns out, totally central to the semantics of quantified DPs, questions, focus, and so on). E.g. $\{\{1,2\},\{1,2,3,4\},\{5,8,9\}\}$. And nothing prevents us from going even higher-order, e.g. to sets of sets of sets and beyond.

Formulae like the one below can be confusing; x has two different senses, depending on where in the formula you're looking.

$$\text{Bad: } \{x : x \in \{x : x \in A\}\}$$

For this reason, it's always best practice to never reuse variables. To change this sort of formula into something reasonable, rename variables from the inside out:

$$\text{Better: } \{x : x \in \{y : y \in A\}\}$$

3 Pairs, relations, functions

3.1 Ordered pairs

Ordered pairs: pairs of two possibly distinct objects, written (x, y).[1]

1 (x, y) can also be given set-theoretically by $\{\{x\}, \{x, y\}\}$. You won't need to know this.

Unsurprisingly (given the name), order matters: $(x, y) \neq (y, x)$. Well, unless....

More generally, n-**tuples**: $(x, y, z, ...)$. 2-tuples are ordered pairs. 3-tuples are triples. There's probably a name for a 4-tuple (but I can't think of it).

Notice that members of tuples can recur: (x, y, x) is distinct from (x, y)

3.2 Relations as sets of pairs

Cartesian products: $A \times B := \{(x, y) : x \in A \text{ and } y \in B\}$

Example: $\{1,3\} \times \{2,4\} = \{(1, 2), (1, 4), (3, 2), (3, 4)\}$

Binary relation: any set of pairs of objects. More precisely, a binary relation R between two sets A and B is a set of pairs whose first entries are elements of A and whose second entries are elements of B.

In this case, A is the **domain** of R, and B is the **codomain** or **range** of R.[2] These notions can be characterized formally as follows:

- $\text{Dom}(R) := \{x : \text{for some } y, (x,y) \in R\}$
- $\text{Ran}(R) := \{y : \text{for some } x, (x,y) \in R\}$

Example: $X = \{$Bill, Carl, Mary, Sue$\}$, and $Y = \{$bike, car, motorcycle, skateboard$\}$, Suppose that Bill rides the bike, Mary rides the motorcycle, and Sam rides the skateboard. Here, the binary relation "rides" is given by:

$$\{(\text{Bill, bike}), (\text{Mary, motorcycle}), (\text{Sam, skateboard})\}$$

Or, equivalently:

$$\{(x, y) : x \text{ rides } y\}$$

Other examples:

i. $\{(x, y) : x < y\}$. What is the domain? The codomain?
ii. $\{(x, y) : x \text{ is the mother of } y\}$. What is the domain? The codomain?

Types of relations (not exhaustive):

i. Reflexive: for any x in R's domain, $(x,x) \in R$
ii. Symmetric: for any x and y if $(x,y) \in R$, then $(y,x) \in R$.
iii. Antisymmetric: for any x, y, if $(x,y) \in R$, and $(y,x) \in R$, then $x = y$.
iv. Transitive: for any x, y, and z, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

2 Codomain and range are not actually totally equivalent, but for our purposes, conflating them is fine.

3.3 Functions

A **function** f is any relation where each input is paired with at most one output. More formally, given any x, there is at most one $(x, y) \in f$.

Because functions are relations, they have domains and ranges. The range of a function is sometimes referred to as its **image**.

Instead of $(x, y) \in f$, we'll write $f(x) = y$.

Notice that while any function always gives the same output for a given input, functions can pair distinct inputs with the same output. E.g.:

$$2^2 = -2^2 = 4$$

Functions (and relations) can be **partial**. Given some x, if a function (or relation) f doesn't include any pair of the form (x, y), then $f(x)$ is **undefined**.

Tabular notation for functions. Here, a partial square root function:

$$\begin{bmatrix} 1 \to 1 \\ 4 \to 2 \\ 9 \to 3 \end{bmatrix}$$

Identity function: a function that maps anything in its domain onto itself, i.e. a set of the form $\{(x, y) : x = y\}$.

The **characteristic functions** of a set A is the function f_A that maps any x in A onto 1 and any y not in A onto 0.

Relations as functions into sets: given that a relation can be defined as a set of pairs, and any set has a characteristic function, relations can be equivalently conceptualized as functions from pairs to 1 and 0:

$$f_R(x,y) \text{ iff } (x,y) \in R$$

4 For next time

We'll see a bit of propositional logic, and we'll see how to use the concepts introduced today to give a *semantics* for propositional logic.

Make sure you're up-to-date with the reading, especially Allwood et al., Ch. 4. Your first homework assignment will be distributed on Wednesday 9/16/15.