Notes on propositional logic

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1 Vocabulary and syntax

Vocabulary (i.e. building blocks):

- Infinite alphabet of *propositional variables*: $p, q, r, \ldots$
- A 'hook' symbol, $\neg$, corresponding to *it's false that*.
- A 'wedge' symbol, $\wedge$, corresponding to *and*.
- A 'vee' symbol, $\vee$, corresponding to inclusive *or*
- A right-facing arrow, $\Rightarrow$, corresponding to (a particular construal of) *if …then …*
- Left and right parentheses, for punctuation: ( )

Syntax: the set of well-formed formulas ('WFF') of propositional logic is defined as follows:

- For any propositional variable v, $v \in \textsf{WFF}$. These are the **atomic propositions**.
- If $\varphi \in \textsf{WFF}$, then $\neg\varphi \in \textsf{WFF}$. These are the **negations**.
- If $\varphi \in \textsf{WFF}$ and $\psi \in \textsf{WFF}$, then $(\varphi \wedge \psi) \in \textsf{WFF}$. These are the **conjunctions**.
- If $\varphi \in \textsf{WFF}$ and $\psi \in \textsf{WFF}$, then $(\varphi \vee \psi) \in \textsf{WFF}$. These are the **disjunctions**.
- If $\varphi \in \textsf{WFF}$ and $\psi \in \textsf{WFF}$, then $(\varphi \Rightarrow \psi) \in \textsf{WFF}$. These are the **(material) conditionals**.
- Nothing else is in WFF. (Can you figure out why this condition is necessary?)

For example, the formulas on the left are in WFF (why?). Those on the right are not (why not?):

$$\begin{array}{cc} \neg(p \wedge q) & p \wedge q \vee r \\ (\neg p \wedge q) & (p \wedge \\ (p \Rightarrow \neg(q \vee r)) & (p \Rightarrow \neg) \end{array}$$

Unlike English, the formulas of propositional logic are *unambiguous*. E.g. (recalling our previous case *eggs and ham or bacon*), $p \wedge q \vee r \notin \textsf{WFF}$, though $(p \wedge (q \vee r)) \in \textsf{WFF}$ and $((p \wedge q) \vee r) \in \textsf{WFF}$.[1]

It's common practice to omit the outermost parentheses from formulas, since doing so never creates ambiguity. I will follow that practice here.

[1] We could write the predicate logic grammar as a context-free phrase structure grammar, if we wished. And we could also represent proofs of syntactic well-formedness as trees, just like in linguistic syntax.

2 Semantics of propositional logic

Any $\varphi \in \textsc{wff}$ has a semantic value relative to an **assignment** of values to the propositional variables, sometimes also known as a **valuation** or **possible world**. We'll write this $[\![\varphi]\!]^w$.

Formally, w is some (total) function from propositional variables into truth values. Here is one possible value for w, presented extensionally using the tabular notation:[2]

$$w \coloneqq \begin{bmatrix} & \vdots & \\ p & \to & 1 \\ q & \to & 0 \\ r & \to & 0 \\ s & \to & 1 \\ & \vdots & \end{bmatrix}$$

For atomic formulas, we have the following rule for interpretation:

$$[\![p]\!]^w \coloneqq w(p)$$

Given the value for w we just defined in the table, $[\![p]\!]^w = w(p) = 1$, $[\![q]\!]^w = w(q) = 0$, etc.

The rules for interpreting complex formulas are given **recursively** as follows (that is, each φ or ψ could itself be an arbitrarily syntactically complex formula):

- $[\![\neg\varphi]\!]^w \coloneqq 1 - [\![\varphi]\!]^w$
- $[\![\varphi \wedge \psi]\!]^w \coloneqq \texttt{Min}([\![\varphi]\!]^w, [\![\psi]\!]^w)$
- $[\![\varphi \vee \psi]\!]^w \coloneqq \texttt{Max}([\![\varphi]\!]^w, [\![\psi]\!]^w)$
- $[\![\varphi \Rightarrow \psi]\!]^w \coloneqq \texttt{Max}([\![\neg\varphi]\!]^w, [\![\psi]\!]^w)$

Besides the notion of arithmetic subtraction in the clause for negation, the definitions refer to `Min` and `Max`. These are just functions from ordered pairs of two truth values to a third truth value, as follows:

$$\underbrace{\begin{bmatrix} (1,1) & \to & 1 \\ (1,0) & \to & 0 \\ (0,1) & \to & 0 \\ (0,0) & \to & 0 \end{bmatrix}}_{\texttt{Min}} \quad \underbrace{\begin{bmatrix} (1,1) & \to & 1 \\ (1,0) & \to & 1 \\ (0,1) & \to & 1 \\ (0,0) & \to & 0 \end{bmatrix}}_{\texttt{Max}}$$

In other words, `Min` takes a pair of two numbers and returns the larger of the two, while `Max` returns the smaller of the two. Another way of putting this: `Min` returns 1 only when **both** numbers are 1, and `Max` returns 1 whenever **at least one** of the numbers is 1.

You should find that this matches the intuitive meanings of $\wedge$ and $\vee$ quite closely. This trick is made possible by treating the range of w as numerical. Other equivalent options are, of course, available (see e.g. the treatment here).

[2]Notice that w's can be regarded as characteristic functions! In other words, we can think of possible worlds as sets of the formulas they regard as true!

E.g., suppose $w(\mathrm{p}) = 1$, $w(\mathrm{q}) = 1$, and $w(\mathrm{r}) = 0$. We calculate $[\![(\mathrm{p} \vee \neg\mathrm{q}) \wedge \mathrm{r}]\!]^w$ like so:

$$\begin{aligned}
[\![(\mathrm{p} \vee \neg\mathrm{q}) \wedge \mathrm{r}]\!]^w &= \mathtt{Min}([\![\mathrm{p} \vee \neg\mathrm{q}]\!]^w, [\![\mathrm{r}]\!]^w) && \text{(Semantic clause for } \wedge\text{)} \\
&= \mathtt{Min}(\mathtt{Max}([\![\mathrm{p}]\!]^w, [\![\neg\mathrm{q}]\!]^w), [\![\mathrm{r}]\!]^w) && \text{(Semantic clause for } \vee\text{)} \\
&= \mathtt{Min}(\mathtt{Max}([\![\mathrm{p}]\!]^w, 1 - [\![\mathrm{q}]\!]^w), [\![\mathrm{r}]\!]^w) && \text{(Semantic clause for } \neg\text{)} \\
&= \mathtt{Min}(\mathtt{Max}(w(\mathrm{p}), 1 - w(\mathrm{q})), w(\mathrm{r})) && \text{(Semantic clause for atomic formulas)} \\
&= \mathtt{Min}(\mathtt{Max}(1, 1 - 1), 0) && \text{(By assumption)} \\
&= \mathtt{Min}(\mathtt{Max}(1, 0), 0) && \text{(Arithmetic)} \\
&= \mathtt{Min}(1, 0) && \text{(By } \mathtt{Max}\text{)} \\
&= 0 && \text{(By } \mathtt{Min}\text{)}
\end{aligned}$$

Exercise: find values for $w(\mathrm{p})$, $w(\mathrm{q})$, and $w(\mathrm{r})$ that yield 1 as a result for $[\![(\mathrm{p} \vee \neg\mathrm{q}) \wedge \mathrm{r}]\!]^w$.

3 Abstracting away from the world

3.1 Truth tables

It is sensible and coherent to talk about the semantic value of any WFF relative to a specific world w. But sometimes we need to take a broader view.

For example, suppose I ask whether p and $\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$ are semantically **equivalent** in propositional logic. That's a question about whether for *every* possible world w, $[\![\mathrm{p}]\!]^w = [\![\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})]\!]^w$:

$$\varphi \text{ and } \psi \text{ are equivalent ('}\mathrm{p} \equiv \mathrm{q}\text{') iff for every } w,\ [\![\varphi]\!]^w = [\![\psi]\!]^w$$

How do we evaluate whether two formulas are equivalent? Given that there's an infinite number of propositional variables, there's likewise an infinite number of possible worlds! Fortunately, we don't need to consider each possible world individually in order to determine equivalence. Instead, we can construct a **truth table**.

For example, suppose we're trying to figure out whether p and $\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$ are equivalent. The way to check this is to see whether there any way to assign values to p and q that delivers different values for p and $\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$. We check this with the following truth table:

p	q	$\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$
1	1	1
0	1	0
1	0	1
0	0	0

The p column and the $\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$ column are *the same*! In other words, for any of the four ways of assigning values to p and q, p and $\mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$ have the same value.[3] Ergo, for any possible world whatsoever, the two formulas have the same value. Ergo, $\mathrm{p} \equiv \mathrm{p} \wedge (\mathrm{p} \vee \mathrm{q})$.

Notice that $\varphi \equiv \psi$ iff for any possible world w, $[\![(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)]\!]^w = 1$. More generally, when $[\![\varphi]\!]^w = 1$ for any possible choice of w, we say that φ is a **tautology**.[4]

[3] In other words, we're now considering *equivalence classes* of possible worlds.

[4] Yes, I now realize this is Greek and not Latin…

The semantics for our connectives can also be characterized in terms of truth tables:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \Rightarrow \psi$
1	1	1	1	1
0	1	0	1	1
1	0	0	1	0
0	0	0	0	1

Truth tables generalize to arbitrarily complex formulas:

φ	ψ	χ	...
1	1	1	...
0	1	1	...
1	0	1	...
0	0	1	...
1	1	0	...
0	1	0	...
1	0	0	...
0	0	0	...

Some famous propositional logical equivalences (Exercise: prove these using truth tables!!):

- **Double-negation elimination**:
  - ▷ $\neg\neg p \equiv p$
- **Distributive laws**:
  - ▷ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - ▷ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **DeMorgan's laws**:
  - ▷ $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  - ▷ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- **Interdefinability of wedge and vee** (follows from DeMorgan's and double-negation elimination):
  - ▷ $p \wedge q \equiv \neg(\neg p \vee \neg q)$
  - ▷ $p \vee q \equiv \neg(\neg p \wedge \neg q)$
- **Eliminability of the material conditional**:
  - ▷ $p \Rightarrow q \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$

Though we've defined three binary connectives ($\wedge, \vee, \Rightarrow$), *any one* of these plus negation is enough to capture the functionality of the other two. In a sense, the other two *needn't be* separately defined (and in practice, they frequently aren't):

any member of $\{(C, \neg) : C \in \{\wedge, \vee, \Rightarrow\}\}$ is **functionally complete**

So, we could make do with just $\wedge$ and $\neg$, with just $\vee$ and $\neg$, or with just $\Rightarrow$ and $\neg$.[5]

[5] It turns out we can actually get by with just a single, binary connective, written '$\uparrow$' and known as the **Sheffer stroke**. It has the following semantics (in prose: '$\varphi \uparrow \psi$' means 'not both φ and ψ'):

$$[\![\varphi \uparrow \psi]\!]^w := 1 - \texttt{Min}([\![\varphi]\!]^w, [\![\psi]\!]^w)$$

For example, $\neg p \equiv p \uparrow p$, $p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q)$, and so on.

3.2 Meanings as sets of possible worlds

Here's another way to abstract away from particular possible worlds. We can consider the semantic values of propositional logic formulas **simpliciter**.

The way we do this is by identifying $[\![\varphi]\!]$ with the set of possible worlds that **satisfy** φ — in other words, the worlds where φ is true. This is just a different perspective on the semantics developed earlier:

$$w \in [\![\varphi]\!] \text{ iff } [\![\varphi]\!]^w = 1$$

We start by giving a meaning for atomic formulas (note the absence of the superscript here):[6]

$$[\![p]\!] \coloneqq \{w : w(p) = 1\}$$

…And then, we give meanings to complex formulas, in terms of *set-theoretic operations*:

- $[\![\neg\varphi]\!] \coloneqq \overline{[\![\varphi]\!]}$
- $[\![\varphi \wedge \psi]\!] \coloneqq [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] \coloneqq [\![\varphi]\!] \cup [\![\psi]\!]$
- $[\![\varphi \Rightarrow \psi]\!] \coloneqq [\![\neg\varphi]\!] \cup [\![\psi]\!]$

From our experience with set theory, we know, e.g., that $w \in [\![\varphi \wedge \psi]\!]$ iff $w \in \varphi$ **and** $w \in \psi$ — i.e., iff both φ and ψ are true at w. This matches up *precisely* with our earlier semantics for conjunction, which also requires φ and ψ to both be true for $\varphi \wedge \psi$ to be true. Mutatis mutandis for the other connectives:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \Rightarrow \psi$
1	1	1	1	1
0	1	0	1	1
1	0	0	1	0
0	0	0	0	1

Because every propositional equivalence holds in the set-theoretic version of the semantics, all of the previous equivalences turn out to have set-theoretic analogs. For example:

- $\overline{\overline{A}} = A$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- …

Next class:

- We will see how to make this semantics more compositional by assigning meanings to the connectives proper. That is, we will be able to talk about $[\![\neg]\!]$, $[\![\wedge]\!]$, $[\![\vee]\!]$, and $[\![\Rightarrow]\!]$, rather than needing to talk about the semantics of formulas containing them.
- This will likewise allow us to move away from the flatter syntax relied on today into one more closely resembling (going analyses of) natural language.

[6]In terms of characteristic functions, $[\![p]\!] \coloneqq \{w : p \in w\}$.