First-order predicate logic (FOPL)

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1 Introducing FOPL by example

We begin with statements of predication (and statements built from statements of predication):

1. Bob is human
   human bob

2. z is French and not German
   $\textit{french}\, z \wedge \neg \textit{german}\, z$

And we can add relations to the mix:

1. Mary saw John
   $\textit{saw}(\textit{mary}, \textit{john})$

2. if x is French, then x didn't show y to Steve
   $\textit{french}\, x \Rightarrow \neg \textit{show}(x, y, \textit{steve})$

We also have quantified statements:

1. Something stinks
   $\exists x.\, \textit{stinks}\, x$

2. z likes everything
   $\forall y.\, \textit{likes}(z, y)$

3. Everyone is French and German
   $\forall z.\, \textit{french}\, z \wedge \textit{german}\, z$

Restricted quantification is built from $\exists$ and $\wedge$, or $\forall$ and $\Rightarrow$ (Exercise: Why *these* pairings? Why not $\exists$ and $\Rightarrow$, or $\forall$ and $\wedge$?):[1]

1. A cat meowed
   $\exists x.\, \textit{cat}\, x \wedge \textit{meowed}\, x$

2. Every cat meowed
   $\forall z.\, \textit{cat}\, z \Rightarrow \textit{meowed}\, z$

3. A linguist likes a philosopher
   $\exists x.\, \textit{linguist}\, x \wedge \exists y.\, \textit{philosopher}\, y \wedge \textit{likes}(x, y)$
   $\exists x.\, \exists y.\, \textit{linguist}\, x \wedge \textit{philosopher}\, y \wedge \textit{likes}(x, y)$

[1] I omit some parentheses because $a \wedge (b \wedge c)$ is equivalent to $(a \wedge b) \wedge c$. Also, similarly to the λ calculus, we will always assume that the scope of $\exists v$ and $\forall v$ extend as far to the right as possible.

We can also represent some cardinality statements:

1. At least two things are blue
   $\exists x.\, \exists y.\, x \neq y \wedge \mathit{blue}\, x \wedge \mathit{blue}\, y$

2. At least three boys are French
   $\exists x.\, \mathit{boy}\, x \wedge \exists y.\, \mathit{boy}\, y \wedge \exists z.\, \mathit{boy}\, z \wedge x \neq y \wedge y \neq z \wedge x \neq z \wedge \mathit{french}\, x \wedge \mathit{french}\, y \wedge \mathit{french}\, z$

2 Negation and duality

Quantified statements can be negated:

1. Nothing is blue (i.e. it's false something is blue)
   $\neg \exists x.\, \mathit{blue}\, x$

2. Not every linguist came:
   $\neg \forall x.\, \mathit{linguist}\, x \Rightarrow \mathit{came}\, x$

And just as $\vee$ and $\wedge$ are DeMorgan duals with respect to negation…

1. $\neg(\varphi \vee \psi) = \neg\varphi \wedge \neg\psi$

2. $\neg(\varphi \wedge \psi) = \neg\varphi \vee \neg\psi$

… So are $\exists$ and $\forall$:

1. $\neg\exists v.\, \varphi = \forall v.\, \neg\varphi$

2. $\neg\forall v.\, \varphi = \exists v.\, \neg\varphi$

So we can represent our first two negated statements equivalently as follows:

1. Nothing is blue (i.e. it's false something is blue)
   $\neg \exists x.\, \mathit{blue}\, x = \forall x.\, \neg \mathit{blue}\, x$

2. Not every linguist came:
   $\neg \forall x.\, \mathit{linguist}\, x \Rightarrow \mathit{came}\, x = \exists x.\, \neg(\mathit{linguist}\, x \Rightarrow \mathit{came}\, x)$
   $= \exists x.\, \mathit{linguist}\, x \wedge \neg \mathit{came}\, x$ [since $\neg(\varphi \Rightarrow \psi) = \neg(\neg\varphi \vee \psi) = \varphi \wedge \neg\psi$]

3 Scope ambiguity

Representing scope ambiguity of the quantifiers and negation:

1. John didn't see a famous linguist
   $\neg\exists x.\, \mathit{famous}\, x \wedge \mathit{linguist}\, x \wedge \mathit{see}(\mathit{john}, x)$
   $\exists x.\, \mathit{famous}\, x \wedge \mathit{linguist}\, x \wedge \neg\mathit{see}(\mathit{john}, x)$

2. Every boy isn't french
   $\forall x.\, \mathit{boy}\, x \Rightarrow \neg\mathit{french}\, x$
   $\neg\forall x.\, \mathit{boy}\, x \Rightarrow \mathit{french}\, x$

(Some of these representations could be given equivalently by appealing to duality w.r.t. negation.)

Notice the placement of the negation with respect to the restriction. What's wrong with the following?

$$\exists x.\, \neg(\mathit{famous}\, x \wedge \mathit{linguist}\, x \wedge \mathit{see}(\mathit{john}, x))$$

Representing scope ambiguities with two quantifiers

1. A linguist met every philosopher
   $\exists x.\, \textit{linguist}\, x \wedge (\forall y.\, \textit{philosopher}\, y \Rightarrow \textit{met}(x, y))$
   $\forall y.\, \textit{philosopher}\, y \Rightarrow (\exists x.\, \textit{linguist}\, x \wedge \textit{met}(x, y))$

Each of these cases has a *surface-scope* interpretation, on which the order of the operators corresponds to their linear order in the English sentence, and an *inverse-scope* interpretation, on which the order of the operators is the reverse of their linear order in the English sentence.

In general, a sentence with n operators (drawn from $\neg$, $\exists$, $\forall$) will have $n!$ scope renderings.

Not all of these interpretations will always be distinct. For example, two quantifiers of the same kind are scopally commutative:

1. $\exists v.\, \exists u.\, \varphi = \exists u.\, \exists v.\, \varphi$

2. $\forall v.\, \forall u.\, \varphi = \forall u.\, \forall v.\, \varphi$

4 FOPL as semantic metalanguage

We can use FOPL to regiment our semantic metalanguage.

For example, here is one way to notate a property that holds of an x iff x saw a linguist:

$$\lambda x.\, \exists y.\, \textit{linguist}\, y \wedge \textit{saw}(x, y)$$

Keep this in mind. We'll be seeing more of it in the coming weeks.

5 Syntax of FOPL

Vocabulary:

- **Terms**: an infinite stock of **variables**: x, y, z, ...
- A collection of n-ary **predicates**: *runs*, *likes*, *gave*, ...
- The propositional logic **connectives**: $\neg$, $\wedge$, $\vee$, $\Rightarrow$. Plus punctuation: ., (, and).
- An **existential quantifier** $\exists$, and a **universal quantifier** $\forall$.

Complex formulas. The WFF of propositional logic is the smallest set such that:

- Predicates applied to the appropriate number of terms are in WFF. E.g., $\textit{left}\, x$, $\textit{saw}(x, y)$, ... These are the atomic formulas.
- If φ is in WFF, then $\neg\varphi$ is in WFF.
- If φ and ψ are in WFF, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \Rightarrow \psi)$ are all in WFF.
- If φ is in WFF, then $(\exists v.\, \varphi)$ and $(\forall v.\, \varphi)$ are in WFF, for any variable v.

As before, we adopt the convention of omitting outermost parentheses. I also like (as above) to omit parentheses when doing so doesn't create ambiguity (cf. fn. 1), but whether you do so is up to you.

6 Semantics of FOPL

As in propositional logic, we need a way to assign values to variables. This time, however, the variables denote things of type $\mathtt{e}$:

- $[\![v]\!]^g = g\,v$

Proper names and predicates will be valued by $[\![\cdot]\!]^g$ in the way we've done in class (though notice that the meaning of relations is given as the characteristic function of a set of ordered pairs):

- $[\![\mathit{bob}]\!]^g = \textsf{B}$
- $[\![\mathit{left}]\!]^g = \lambda x.\, \textsf{LEFT}\, x$
- $[\![\mathit{likes}]\!]^g = \lambda(x, y).\, \textsf{LIKES}\,(x, y)$
- …

Predications are evaluated by finding the values of the predicate and its arguments, and then applying the former to the latter:

- $[\![P\,v]\!]^g = [\![P]\!]^g\,[\![v]\!]^g$
- $[\![R\,(v, u)]\!]^g = [\![R]\!]^g\,([\![v]\!]^g, [\![u]\!]^g)$
- …

The meanings of the connectives are unchanged from propositional logic:

- $[\![\neg\varphi]\!]^g = 1 - [\![\varphi]\!]^g$
- $[\![\varphi \wedge \psi]\!]^g = \mathtt{Min}\,\{[\![\varphi]\!]^g, [\![\psi]\!]^g\}$
- $[\![\varphi \vee \psi]\!]^g = \mathtt{Max}\,\{[\![\varphi]\!]^g, [\![\psi]\!]^g\}$
- $[\![\varphi \Rightarrow \psi]\!]^g = \mathtt{Max}\,\{[\![\neg\varphi]\!]^g, [\![\psi]\!]^g\}$

Finally, the meanings of the quantifiers rely on *assignment modification*:

- $[\![\exists v.\, \varphi]\!]^g = \mathtt{Max}\,\{[\![\varphi]\!]^{g[v \to x]} : x \in \mathtt{e}\}$
- $[\![\forall v.\, \varphi]\!]^g = \mathtt{Min}\,\{[\![\varphi]\!]^{g[v \to x]} : x \in \mathtt{e}\}$

Relies on **minimal assignment modification**:

- $g[v \to x]$ is the assignment h such that $h\,v = x$, and for any $u \neq v$, $h\,u = g\,u$.

Mnemonically, you can think of $g[v \to x]$ as "the assignment mapping v to x, but otherwise just like g".

Essentially, existential quantification is like a huge disjunction (if a **or** b **or** c **or** … makes φ true, then $\exists v.\, \varphi$ is true), and universal quantification is like a huge conjunction (if a **and** b **and** c **and** … make φ true, then $\forall v.\, \varphi$ is true).