

Fact 4 (Isomorphism of lawful lifts) is false (a counterexample is provided below). A weaker result does hold: all lawful lifts are *observationally equivalent*, in that their interpretations of any formula project down to identical substrate meanings:

Fact 4’ (Observational equivalence). For any two lawful lifts over the same substrate δ and any $\phi \in \mathcal{L}$, $\langle \phi \rangle_1^{\downarrow 1} = \langle \phi \rangle_2^{\downarrow 2} \cdot^\dagger$

Proof. Let ϕ' be the result of repeatedly replacing subformulas $\neg\neg\psi$ of ϕ with ψ . By Fact 1.ii, $\langle \phi \rangle_k = \langle \phi' \rangle_k$ for all k . As ϕ' is $\neg\neg$ -free, $\langle \phi \rangle_1^{\downarrow 1} = [\phi'] = \langle \phi \rangle_2^{\downarrow 2}$ by Fact 2.ii. $\square$

Counterexample to Fact 4. Fix DPL as the substrate. Let A be atomic, $a := [A]$, and $\mathbf{1}$ a $\neg\neg$ -free static tautology, as in the proof of Fact 3. Since a is static, $\neg_\delta \neg_\delta a = a$, and for any k , $\langle \neg A \wedge \mathbf{1} \rangle_k = (\neg_\delta a)^{\uparrow k}$ and $\langle \neg A \rangle_k = \sim_k a^{\uparrow k}$ (cf. the remark at the end of Section 5 that ϕ and $\phi \wedge \mathbf{1}$ are not always equivalent). In Instance 1, where $m^{\uparrow 1} = (m, \neg_\delta m)$ and $\sim_1(m, n) = (n, m)$, we have the following (likewise for Instances 2 and 3):

$$\langle \neg A \wedge \mathbf{1} \rangle_1 = (\neg_\delta a)^{\uparrow 1} = (\neg_\delta a, a) = \sim_1 a^{\uparrow 1} = \langle \neg A \rangle_1.$$

In Instance 4, the canonical lift, where $m^{\uparrow 4} = (\top, m)$ and $\sim_4(b, m) = (\bar{b}, m)$, we have:

$$\langle \neg A \wedge \mathbf{1} \rangle_4 = (\neg_\delta a)^{\uparrow 4} = (\top, \neg_\delta a) \neq (\top, a) = \sim_4 a^{\uparrow 4} = \langle \neg A \rangle_4.$$

No function f can satisfy $f(\langle \phi \rangle_1) = \langle \phi \rangle_4$ for all ϕ : the single value $\langle \neg A \wedge \mathbf{1} \rangle_1 = \langle \neg A \rangle_1$ would have to map to $\langle \neg A \wedge \mathbf{1} \rangle_4$ and $\langle \neg A \rangle_4$. (Instance 4 is minimal in the sense of Fact 3, but not in the sense of drawing the fewest distinctions among lifted meanings.) Note, however, that $(\top, \neg_\delta a)^{\downarrow 4} = \neg_\delta a = (\top, a)^{\downarrow 4}$. While the two values are distinguishable, they are never, in practice, distinguished (e.g., $\downarrow$ applies to conjuncts inside $\langle \phi \rangle$ and is used to characterize truth and entailment for lifted interpretations).

More generally, if there are $\neg\neg$ -free ϕ, ψ , neither of the form $\neg\chi$, with $[\phi] = \neg_\delta[\psi]$ and $[\psi] = \neg_\delta[\phi]$, then lawful lifts can disagree about whether $\langle \phi \rangle = \langle \neg\psi \rangle$. Instances 1–3 (and Instance 5) identify them. Instance 4 does not, and so Fact 4 fails. The error in the published proof is in showing that f is well defined: for $m^{\uparrow 1} = \sim_1 n^{\uparrow 1}$ (as with ϕ, ψ above), f ’s clauses yield $m^{\uparrow 2}$ and $\sim_2 n^{\uparrow 2}$, which the proof (circularly) assumes coincide.

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Appeals to Fact 4 should be read as appeals to Fact 4’ (e.g., the abstract’s claim that all such lifted systems are isomorphic should read “observationally equivalent”). The exception is footnote 9, whose claim that canonical  $\Delta$ ’s are isomorphic does not follow.

The paper’s claims about isomorphisms between existing accounts do stand. The published proof is valid for any two lifts that agree on which  $m, n$  satisfy  $m^{\uparrow} = \sim n^{\uparrow}$ , and

$\dagger$  This is the Corollary at the end of Appendix A, where it was derived from Fact 4.

so in particular for any lifts obeying the following law: if  $m = \neg_\delta n$  and  $n = \neg_\delta m$ , then  $m^\uparrow = \sim n^\uparrow$  (the converse follows from Emb, Neg, and Inv). Equivalently: if  $\neg_\delta \neg_\delta n = n$ , then  $(\neg_\delta n)^\uparrow = \sim n^\uparrow$ . Instances 1–3 and 5 obey this law, and so are isomorphic; Instance 4 does not. Adding it to Definition 3 would restore Fact 4 (and disallow Instance 4).